Generalized Inverse of Tensors and application in solving $\mathscr{A} *_M \mathscr{X} = \mathscr{B}$ under M-product

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Outline

Tensor Representations

For a third order tensor $\mathscr{A} = (a_{ijk})$, $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq k \leq p$

- The ith frontal slice of a tensor $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$ is denoted by $\mathscr{A}^{(i)} = \mathscr{A}(:,:,i)$.
- The tube fibers of $\mathscr{A}$ are labeled with either $\mathscr{A}(i,j,:)$ or $\mathscr{A}(i,:,k)$ or $\mathscr{A}(:,j,k)$.
- Elements of $\mathscr{A}$ are denoted either by $(\mathscr{A})_{ijk}$ or a_{ijk}, such that $i = \overline{1,m}$, $j = \overline{1,n}$ and $k = \overline{1,p}$.

3-Mode Product (Tensor-Matrix multiplication)

Definition

Let $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$ be a tensor and $M \in \mathbb{R}^{p \times p}$ be a matrix. The 3-mode product of $\mathscr{A}$ with M is denoted by $\mathscr{A} \times_3 M \in \mathbb{R}^{m \times n \times p}$ and element-wise defined as

$$(\mathscr{A} \times_3 B)_{ijk} = \sum_{s=1}^{p} a_{ijs} b_{ks} \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \; k = 1, 2 \ldots, p.$$

3-Mode Product (Tensor-Matrix multiplication)

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$$(\mathscr{A} \times_3 B)_{ijk} = \sum_{s=1}^{p} a_{ijs} b_{ks} \quad i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,\ k = 1,2\ldots,p.$$

Note: From here onwards we will assume M to be invertible and $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$.

☞ Kolda, Tamara G., and Brett W. Bader. Tensor decompositions and applications. SIAM review. 2009; 51(3):455-500.

Face-wise Product

Tensor-Tensor Multiplication:

DEFINITION

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $B \in \mathbb{R}^{n\times k\times p}$ be two tensors. The face-wise product of $\mathscr{A}$ and $\mathscr{B}$ is denoted by $\mathscr{A} \triangle \mathscr{B} \in \mathbb{R}^{m\times k\times p}$ and element-wise defined as

$$(\mathscr{A} \triangle \mathscr{B})(:,:,i) = \mathscr{A}(:,:,i)\mathscr{B}(:,:,i), \; i = 1,2\ldots,p.$$

☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

M-Product

Tensor-Tensor Multiplication:

Definition

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $B \in \mathbb{R}^{n\times k\times p}$ be two tensors and $M \in \mathbb{R}^{p\times p}$. The M-product of $\mathscr{A}$ and $\mathscr{B}$ is denoted by $\mathscr{A} *_M \mathscr{B} \in \mathbb{R}^{m\times k\times p}$ and defined as

$$\mathscr{A} *_M \mathscr{B} = [(\mathscr{A} \times_3 M)\triangle(\mathscr{B} \times_3 M)] \times_3 M^{-1}.$$

☞ E. Kernfeld, M. Kilmer, S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl. 2015; 485:545-570.

Tensor computations

Definition (Multirank, Tubal rank, tubal Norm)

Let $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$ and $M \in \mathbb{R}^{p \times p}$. Then the

(i) tubal norm of $\mathscr{A}$ is defined as $\|\mathscr{A}\|_M = \max\limits_{1 \leq i \leq p}(\|\tilde{\mathscr{A}}(:,:,i)\|)$, where $\|A\|$ is the norm of a matrix A.

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(i) tubal norm of $\mathscr{A}$ is defined as $\|\mathscr{A}\|_M = \max\limits_{1\leq i\leq p}(\|\tilde{\mathscr{A}}(:,:,i)\|)$, where $\|A\|$ is the norm of a matrix A.

(ii) multirank $\mathscr{A}$ is denote by $r_M(\mathscr{A})$ and defined as $r_M(\mathscr{A}) = (r_1, r_2, \ldots, r_p)$ where $r_i = rank(\tilde{\mathscr{A}}(:,:,i))$, $i = 1, 2, \ldots, p$.

Tensor computations

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(ii) multirank $\mathscr{A}$ is denote by $r_M(\mathscr{A})$ and defined as $r_M(\mathscr{A}) = (r_1, r_2, \ldots, r_p)$ where $r_i = rank(\tilde{\mathscr{A}}(:,:,i))$, $i = 1, 2, \ldots, p$.

(iii) tubal rank of $\mathscr{A}$ is defined by $\text{rank}_M(\mathscr{A}) = \max_{1\leq i\leq p} \text{rank}(\tilde{\mathscr{A}}(:,:,i))$

Tensor computations

Definition (Transformation)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $M \in \mathbb{R}^{p\times p}$. Then $\texttt{mat} : \mathbb{R}^{m\times n\times p} \mapsto \mathbb{R}_M^{mp\times np}$ is defined as

$$\texttt{mat}(\mathscr{A}) = \begin{bmatrix} \tilde{\mathscr{A}}(:,:,1) & 0 & \cdots & 0 \\ 0 & \tilde{\mathscr{A}}(:,:,2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\mathscr{A}}(:,:,p) \end{bmatrix}.$$

Tensor computations

Definition (Transformation)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$, $M \in \mathbb{R}^{p\times p}$. Then $\texttt{mat} : \mathbb{R}^{m\times n\times p} \mapsto \mathbb{R}_M^{mp\times np}$ is defined as

$$\texttt{mat}(\mathscr{A}) = \begin{bmatrix} \tilde{\mathscr{A}}(:,:,1) & 0 & \cdots & 0 \\ 0 & \tilde{\mathscr{A}}(:,:,2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\mathscr{A}}(:,:,p) \end{bmatrix}.$$

The inverse operation $\texttt{mat}^{-1}$ can be defined as follows:

Input $A \in \mathbb{R}^{mp\times np}$ and $M \in \mathbb{R}^{p\times p}$
for $i \leftarrow 1$ to p **do**
$B(:,:,i) = A((i-1)m+1 : im, (i-1)n+1 : in)$
end for
Compute $\texttt{mat}^{-1}(A) = \mathscr{B} \times_3 M^{-1}$

Tensor computations

Every tensor $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$ can be represented as follows:

$$\mathscr{A} = \mathtt{mat}^{-1}(\mathtt{mat}(\mathscr{A})).$$

$$\mathscr{A} *_{\mathbf{M}} \mathscr{B} = \mathtt{mat}^{-\mathbf{1}}(\mathtt{mat}(\mathscr{A})\mathtt{mat}(\mathscr{B})).$$

Tensor computations

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$$\mathscr{A} = \mathtt{mat}^{-1}(\mathtt{mat}(\mathscr{A})).$$

$$\mathscr{A} *_{\mathbf{M}} \mathscr{B} = \mathtt{mat}^{-\mathbf{1}}(\mathtt{mat}(\mathscr{A})\mathtt{mat}(\mathscr{B})).$$

A Common properties for tensors

- Turn tensor $\mathscr{A}$ into a matrix A and draw conclusions about tensor $\mathscr{A}$ based on what is learned about matrix A but this process some time time consuming or tedious.
- Many properties can be considered based on the frontal slices of $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$.

Tensor computations

DEFINITION

[a] The tensor $\mathscr{A} \in \mathbb{R}^{m \times n \times p}$ is called diagonally dominant with respect to $M \in \mathbb{R}^{p \times p}$ if all the frontal slices of $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$ are diagonally dominant.

[a] E. Kernfeld, M. Kilmer, and S. Aeron. Tensor-tensor products with invertible linear transforms. Linear Algebra Appl., 485:545-570, 2015.

Tensor computations

Definition

Consider $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. Then $\mathscr{A}$ is called

(i) diagonally dominant (strictly diagonally dominant) if $\tilde{\mathscr{A}}(:,:,i)$ is diagonally dominant (strictly diagonally dominant) for all i, $i = \overline{1,p}$.

(ii) hermitian positive definite (HPD) if $\tilde{\mathscr{A}}(:,:,i)$ is HPD for all i, $i = \overline{1,p}$.

(iii) nonnegative (denoted by $\mathscr{A} \geq 0$) if

$$(\tilde{\mathscr{A}})_{ijk} \geq 0 \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq p.$$

M-PRODUCT(CONTINUED)

DEFINITION

Let $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$. If $\mathscr{X} \neq \mathscr{O} \in \mathbb{R}^{m\times 1\times n}$ satisfy

$$\mathscr{A} *_M \mathscr{X} = \lambda \mathscr{X}, \;\; \lambda \in \mathbb{R}.$$

Such λ is termed as an M-eigenvalue of $\mathscr{A}$ and $\mathscr{X}$ is the M-eigenvector of $\mathscr{A}$ based on M and λ. Further, the spectral radius of $\mathscr{A}$ is denoted as $\rho(\mathscr{A})$ and is defined as $\rho(\mathscr{A}) = \max\limits_{1\leq i\leq p}\{\rho(\tilde{\mathscr{A}}(:,:,i)\}$.

DEFINITION

The range and null space of $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ relative to $M \in \mathbb{R}^{p\times p}$ are defined, respectively, as

$$\begin{aligned}
\mathscr{R}_M(\mathscr{A}) &= \{\mathscr{A} *_M \mathscr{Z} \;:\; \mathscr{Z} \in \mathbb{R}^{n\times 1\times p}\} \subseteq \mathbb{R}^{m\times 1\times p}, \\
\mathscr{N}_M(\mathscr{A}) &= \{\mathscr{Y} \;:\; \mathscr{A} *_M \mathscr{Y} = \mathscr{O} \in \mathbb{R}^{m\times 1\times p}\} \subseteq \mathbb{R}^{n\times 1\times p}.
\end{aligned}$$

DRAWBACK OF MATRIX -STRUCTURED COMPUTATIONS

TABLE: Comparison of mean CPU time for computing $\mathscr{A}^{-1}$

Size of $\mathscr{A}$	$\text{MT}_{\textit{tensor}}$	Size of $\texttt{mat}(\mathscr{A})$	$\text{MT}_{\texttt{mat}}$ and $\texttt{mat}^{-1}$
$60 \times 60 \times 60$	0.24	3600×3600	15.34
$80 \times 80 \times 80$	0.54	6400×6400	54.27
$100 \times 100 \times 100$	1.05	10000×10000	185.67
$120 \times 120 \times 120$	1.87	14400×14400	532.43

☞ J.K. Sahoo, S. K. Panda, R. Behera, and P. S. Stanimirovic. Computation of tensors generalized inverses under *M*-product and applications. Journal of Mathematical Analysis & Applications, 542(1), 2025.

DRAWBACK OF MATRIX -STRUCTURED COMPUTATIONS

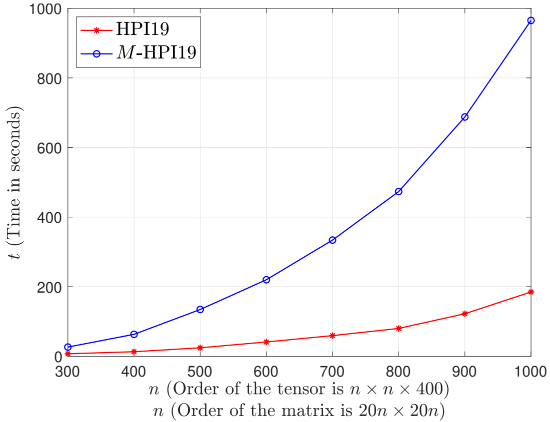

FIGURE: Comparison analysis of mean CPU time for computing the inverse of tensors $\mathscr{A}$ and matrices A

☞ R. Behera, K. Panigrahy, J. K. Sahoo, and Y. Wei. M-QR decomposition and hyperpower iterative (HPI) methods for computing outer inverses of tensors. arXiv preprint, 2024.

Outline

Generalized Inverse of a Tensor

Definition (Moore-Penrose inverse)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $M \in \mathbb{R}^{m\times n}$. If a tensor $\mathscr{X} \in \mathbb{R}^{n\times m\times p}$ satisfies the following properties

- $\mathscr{A} *_M \mathscr{X} *_M \mathscr{A} = \mathscr{A}$
- $\mathscr{X} *_M \mathscr{A} *_M \mathscr{X} = \mathscr{X}$
- $(\mathscr{A} *_M \mathscr{X})^T = \mathscr{A} *_M \mathscr{X}$
- $(\mathscr{X} *_M \mathscr{A})^T = \mathscr{X} *_M \mathscr{A}$,

then $\mathscr{X}$ is called the Moore-Penrose inverse of $\mathscr{A}$ and denoted by $\mathscr{A}^\dagger$.

GENERALIZED INVERSE OF A TENSOR

DEFINITION (MOORE-PENROSE INVERSE)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $M \in \mathbb{R}^{m\times n}$. If a tensor $\mathscr{X} \in \mathbb{R}^{n\times m\times p}$ satisfies the following properties

- $\mathscr{A} *_M \mathscr{X} *_M \mathscr{A} = \mathscr{A}$
- $\mathscr{X} *_M \mathscr{A} *_M \mathscr{X} = \mathscr{X}$
- $(\mathscr{A} *_M \mathscr{X})^T = \mathscr{A} *_M \mathscr{X}$
- $(\mathscr{X} *_M \mathscr{A})^T = \mathscr{X} *_M \mathscr{A}$,

then $\mathscr{X}$ is called the Moore-Penrose inverse of $\mathscr{A}$ and denoted by $\mathscr{A}^\dagger$.

- Tensor generalized inverses have significantly impacted the numerical multilinear algebra, specifically solving multilinear systems, which are obtained from mathematical models.

☞ L. Sun, B. Zheng, C. Bu, and Y. Wei. Moore-Penrose inverse of tensors via Einstein product. Linear and Multilinear Algebra 64(4), (2016):686-698.

☞ R. Behera, J.K. Sahoo, R. N. Mohapatra, and M. Z. Nashed. Computation of generalized inverses of tensors via t-product. Numerical Linear Algebra with Applications 29(2), (2022): e2416.

☞ H. Jin, S. Xu, Y. Wang, and X. Liu. The Moore-Penrose inverse of tensors via the M-product. Computational and Applied Mathematics 42(6), (2023): 294.

Moore-Penrose inverse

Proposition

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. Then

$$\mathscr{A}^{\dagger} = \mathtt{mat}^{-1}(\mathtt{mat}(\mathscr{A})^{\dagger}).$$

Moore-Penrose inverse

Proposition

Let $M \in \mathbb{R}^{p\times p}$ *and* $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. *Then*

$$\mathscr{A}^{\dagger} = mat^{-1}(mat(\mathscr{A})^{\dagger}).$$

Algorithm 2: Computing the Moore-Penrose inverse under M-product

```
procedure MPI(𝒜†)
    Input 𝒜 ∈ ℝ^{m×n×p} and M ∈ ℝ^{p×p}.
    Compute 𝒜̃ = 𝒜 ×_3 M
    for i ← 1 to p do
    𝒵(:,:,i) = (𝒜̃(:,:,i))†
    end for
    Compute 𝒳 = 𝒵 ×_3 M^{-1}
    return 𝒜† = 𝒳
end procedure
```

Drawback of Matrix -Structured Computations

TABLE: Comparison of mean CPU time for computing $\mathscr{A}^{\dagger}$

Size of $\mathscr{A}$	MT_{tensor}	Size of $\mathtt{mat}(\mathscr{A})$	$MT_{\mathtt{mat}}$ and $\mathtt{mat}^{-1}$
$60 \times 80 \times 60$	0.13	3600×4800	11.23
$80 \times 60 \times 80$	0.25	6400×4800	33.41
$100 \times 120 \times 100$	0.65	10000×12000	145.37
$120 \times 150 \times 100$	1.24	12000×15000	376.93

Drazin inverse under M-product

Definition

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ with tubal index k. The Drazin inverse $\mathscr{A}^D$ of $\mathscr{A}$ is the unique tensor r $\mathscr{X} \in \mathbb{R}^{m\times m\times p}$ satisfying $\mathscr{X} *_M \mathscr{A} *_M \mathscr{X} = \mathscr{X}$, $\mathscr{A} *_M \mathscr{X} = \mathscr{X} *_M \mathscr{A}$ and $\mathscr{X} *_M \mathscr{A}^{k+1} = \mathscr{A}^k$.

We can also compute the Drazin inverse using $\mathtt{mat}$ and $\mathtt{mat}^{-1}$, as stated below.

Proposition

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ with $ind(\mathtt{mat}(\mathscr{A})) = k$. Then,

$$\mathscr{A}^D = \mathtt{mat}^{-1}(\mathtt{mat}(\mathscr{A})^D).$$

DRAZIN INVERSE UNDER M-PRODUCT

Algorithm 3: Computing the Drazin inverse under M-product

procedure DRAZIN INVERSE($\mathscr{A}^D$)
    Input $\mathscr{A} \in \mathbb{R}^{m \times m \times p}$ and $M \in \mathbb{R}^{p \times p}$.
    Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$
    for $i \leftarrow 1$ to p **do**
    $k_i = ind(\tilde{\mathscr{A}}(:,:,i))$
    end for
    Compute $k = \max_{1 \leq i \leq p} k_i$
    for $i \leftarrow 1$ to p **do**
    $Z(:,:,i) = (\tilde{\mathscr{A}}(:,:,i))^D$
    end for
    Compute $\mathscr{X} = Z \times_3 M^{-1}$
    return $\mathscr{A}^D = \mathscr{X}$
end procedure

Drazin inverse under the M-product

Example

Let $\mathscr{A} \in \mathbb{R}^{3\times3\times3}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 4 & -4 & -1 \\ -7 & -8 & 7 \\ -1 & -2 & 0 \end{bmatrix}, \ \mathscr{A}(:,:,2) = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 4 & -4 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathscr{A}(:,:,3) = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 4 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We evaluate the tubal index $k = 2 = \max\{1, 1, 2\}$ since $ind(\tilde{\mathscr{A}}(:,:,1)) = 1 = ind(\tilde{\mathscr{A}}(:,:,2)), ind(\tilde{\mathscr{A}}(:,:,3)) = 2.$

Example

By Algorithm 3, we calculate $\mathscr{X} = \mathscr{A}^D$, where

$$\mathscr{X}(:,:,1) = \begin{bmatrix} -11 & -2 & 1 \\ -2 & 1 & 1 \\ -4.5 & -1.5 & -1 \end{bmatrix},$$

$$\mathscr{X}(:,:,2) = \begin{bmatrix} -6 & 1 & -1 \\ 0.5 & -0.5 & -1 \\ 2.75 & -0.75 & 1 \end{bmatrix},$$

$$\mathscr{X}(:,:,3) = \begin{bmatrix} -4 & 1 & 0 \\ 1.5 & -0.5 & 0 \\ 1.75 & -0.75 & 0 \end{bmatrix}.$$

Drazin inverse

A comparison of the mean CPU time (MT) for using tubal index and `mat` operation is presented in Table 3.

Table: Comparison of mean CPU times for computing $\mathscr{A}^D$

Size of $\mathscr{A}$	k	MT (Using tubal index)	Size of $\mathtt{mat}(\mathscr{A})$	$ind(\mathtt{mat}(\mathscr{A}))$	MT (Using $\mathtt{mat}$ and $\mathtt{mat}^{-1}$)
$60 \times 60 \times 60$	1	0.19	3600×3600	1	8.10
$80 \times 80 \times 80$	1	0.37	6400×6400	1	39.37
$100 \times 100 \times 100$	2	0.94	10000×10000	2	169.72
$120 \times 120 \times 120$	2	1.60	14400×14400	2	434.46

Drazin inverse

TABLE: Computational time for computing $\mathscr{A}^D$ for different tensor products

Size of $\mathscr{A}$	k	MT^t	MT^c	MT^M
$300 \times 300 \times 300$	1	34.18	14.14	11.02
$400 \times 400 \times 400$	1	50.80	29.46	28.18
$300 \times 300 \times 300$	2	35.09	16.26	15.75
$400 \times 400 \times 400$	2	51.72	38.93	38.92

Core-EP inverse under M-product

Definition

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ with tubal index k. If a tensor $\mathscr{X} \in \mathbb{R}^{m\times m\times p}$ satisfies $\mathscr{X} *_M \mathscr{A}^{k+1} = \mathscr{A}^k$, $\mathscr{A} *_M \mathscr{X}^2 = \mathscr{X}$ and $(\mathscr{A} *_M \mathscr{X})^* = \mathscr{A} *_M \mathscr{X}$ then $\mathscr{X}$ is called the core-EP inverse of $\mathscr{A}$ and denoted by $\mathscr{A}^{\textcircled{\dagger}}$.

CORE-EP INVERSE UNDER M-PRODUCT

DEFINITION

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ with tubal index k. If a tensor $\mathscr{X} \in \mathbb{R}^{m\times m\times p}$ satisfies $\mathscr{X} *_M \mathscr{A}^{k+1} = \mathscr{A}^k$, $\mathscr{A} *_M \mathscr{X}^2 = \mathscr{X}$ and $(\mathscr{A} *_M \mathscr{X})^* = \mathscr{A} *_M \mathscr{X}$ then $\mathscr{X}$ is called the core-EP inverse of $\mathscr{A}$ and denoted by $\mathscr{A}^{\textcircled{\dagger}}$.

PROPOSITION

Let $M \in \mathbb{R}^{p\times p}$ and $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ with $ind(\mathtt{mat}(\mathscr{A})) = k$. Then

$$\mathscr{A}^{\textcircled{\dagger}} = \mathtt{mat}^{-1}(\mathtt{mat}(\mathscr{A})^{\textcircled{\dagger}}).$$

Core-EP inverse under M-product

Algorithm 4: Core-EP inverse under M-product

```
procedure CORE-EP INVERSE(𝒜^ⓣ)
    Input 𝒜 ∈ ℝ^{m×m×p} and M ∈ ℝ^{p×p}.
    Compute 𝒜̃ = 𝒜 ×₃ M
    for i ← 1 to p do
    k_i = ind(𝒜̃(:, :, i))
    end for
    Compute k = max_{1≤i≤p} k_i
    for i ← 1 to p do
    Z(:, :, i) = (𝒜̃(:, :, i))^ⓣ
    end for
    Compute 𝒳 = Z ×₃ M^{-1}
    return 𝒜^ⓣ = 𝒳
end procedure
```

procedure CORE-EP INVERSE($\mathscr{A}^{\textcircled{\dagger}}$)
- **Input** $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$ and $M \in \mathbb{R}^{p\times p}$.
- Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$
- **for** $i \leftarrow 1$ to p **do**
- $k_i = ind(\tilde{\mathscr{A}}(:,:,i))$
- **end for**
- Compute $k = \max_{1\le i\le p} k_i$
- **for** $i \leftarrow 1$ to p **do**
- $Z(:,:,i) = (\tilde{\mathscr{A}}(:,:,i))^{\textcircled{\dagger}}$
- **end for**
- Compute $\mathscr{X} = Z \times_3 M^{-1}$
- **return** $\mathscr{A}^{\textcircled{\dagger}} = \mathscr{X}$

end procedure

Core-EP inverse under M-product

Example

Let $\mathscr{A} \in \mathbb{R}^{3\times 3\times 3}$ with entries

$$\mathscr{A}(:,:,1) = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 0 & 0 \\ -2 & 2 & -1 \end{bmatrix},\ \mathscr{A}(:,:,2) = \begin{bmatrix} 0 & -2 & -4 \\ -8 & 7 & 0 \\ 11 & -10 & 8 \end{bmatrix},$$

$$\mathscr{A}(:,:,3) = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}, M = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}.$$

Since $ind(\tilde{\mathscr{A}}(:,:,1)) = 1,\ ind(\tilde{\mathscr{A}}(:,:,2)) = 2$ and $ind(\tilde{\mathscr{A}}(:,:,3)) = 3$, the tubal index of $\mathscr{A}$ is equal to $k = 3 = \max\{1,2,3\}$.

CORE-EP INVERSE UNDER M-PRODUCT

EXAMPLE

Based on Algorithm 4, we calculate $\mathscr{X} = \mathscr{A}^{\textcircled{\dagger}}$, where

$$\mathscr{X}(:,:,1) = \begin{bmatrix} 0.0714 & -0.1429 & -0.2143 \\ -0.1429 & 0.2857 & 0.4286 \\ -0.2143 & 0.4286 & 0.6429 \end{bmatrix},$$

$$\mathscr{X}(:,:,2) = \begin{bmatrix} -0.3158 & 0.2925 & 0.6525 \\ -0.3417 & 0.0474 & 0.2991 \\ -0.0620 & -0.2299 & -0.2230 \end{bmatrix},$$

$$\mathscr{X}(:,:,3) = \begin{bmatrix} 0.2619 & -0.1905 & -0.4524 \\ 0.4762 & -0.2857 & -0.7619 \\ 0.2143 & -0.0952 & -0.3095 \end{bmatrix}.$$

CORE-EP INVERSE UNDER M-PRODUCT

A comparison of the mean CPU time (MT) for using the tubal index and `mat` operation is provided in Table 5

TABLE: Comparison of mean CPU time for computing $\mathscr{A}^{\textcircled{\dagger}}$

Size of $\mathscr{A}$	k	MT (Using tubal index)	Size of $\mathtt{mat}(\mathscr{A})$	$ind(\mathtt{mat}(\mathscr{A}))$	MT (Using `mat` and $\mathtt{mat}^{-1}$)
$60 \times 60 \times 60$	1	0.20	3600×3600	1	10.70
$80 \times 80 \times 80$	1	0.39	6400×6400	1	47.58
$100 \times 100 \times 100$	2	1.25	10000×10000	2	217.78
$120 \times 120 \times 120$	2	2.06	14400×14400	2	576.88

CORE-EP INVERSE UNDER M-PRODUCT

TABLE: Computational time for computing $\mathscr{A}^{\textcircled{\dagger}}$ for different tensor products

Size of $\mathscr{A}$	k	MT^t	MT^c	MT^M
$300 \times 300 \times 300$	1	26.26	15.14	15.06
$400 \times 400 \times 400$	1	58.75	39.32	39.11
$300 \times 300 \times 300$	2	25.36	18.74	18.69
$400 \times 400 \times 400$	2	58.78	47.55	46.09

Outline

Higher order Jacobi Method

Algorithm 5: Higher order Jacobi Method based on M-product

procedure JACOBI($\mathscr{A}, \mathscr{B}, \varepsilon$, MAX)
  Input $\mathscr{A} \in \mathbb{R}^{m\times m\times p}$, $\mathscr{B} \in \mathbb{R}^{m\times 1\times p}$ and $M \in \mathbb{R}^{p\times p}$.
  Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$
  for $i = 1$ to p **do**
  Compute $\tilde{\mathscr{D}}(:,:,i) = diag(\tilde{\mathscr{A}}(:,:,i))$, $\tilde{\mathscr{F}}(:\,.\,:,i) = \tilde{\mathscr{A}}(:,:,i) - \tilde{\mathscr{D}}(:,:,i)$
  Compute $\tilde{\mathscr{T}}(:,:,i) = -(\tilde{\mathscr{D}})^{-1}(:,:,i)\tilde{\mathscr{F}}(:\,.\,:,i)$ and $\mathscr{C}(:,1,i) = (\tilde{\mathscr{D}})^{-1}(:,:,i)\mathscr{B}(:,1,i)$
  Initial guess $\tilde{\mathscr{X}}^0(:,1,i)$
  for $s = 1$ to MAX **do**
  $\tilde{\mathscr{X}}^s(:,1,i) = \tilde{\mathscr{T}}(:,:,i)\tilde{\mathscr{X}}^{s-1}(:,1,i) + \mathscr{C}(:,1,i)$
  if $\|\tilde{\mathscr{X}}^s(:,1,i) - \tilde{\mathscr{X}}^0(:,1,i)\| \leq \varepsilon$ **then**
  break
  end if
  $\tilde{\mathscr{X}}^0(:,1,i) \leftarrow \tilde{\mathscr{X}}^s(:,1,i)$
  end for
  end for
  Compute $\mathscr{X}^s = \tilde{\mathscr{X}}^s \times_3 M^{-1}$
  return $\mathscr{X}^s$
end procedure

Higher order Jacobi Method

TABLE: Comparison analysis of CPU-time, residual errors for Jacobi method for different order tensors and matrices with taking $\varepsilon = 10^{-10}$

Size of $\mathscr{A}$	IT^M	MT^M	Size of A	IT	MT
$100 \times 100 \times 400$	88	0.26	2000×2000	96	43.04
$200 \times 200 \times 400$	88	0.81	4000×4000	101	71.56
$300 \times 300 \times 400$	89	1.01	6000×6000	93	8275
$400 \times 400 \times 400$	89	1.80	8000×8000	99	19466
$500 \times 500 \times 400$	89	2.36	10000×10000	99	34732

Higher order Gauss-Seidel Method

Algorithm 6: Higher order Gauss-Seidel method based on M-product

procedure GAUSS-SEIDEL($\mathscr{A}, \mathscr{B}, \varepsilon$, MAX)
  Input $\mathscr{A} \in \mathbb{R}^{m \times m \times p}, \mathscr{B} \in \mathbb{R}^{m \times 1 \times p}$ and $M \in \mathbb{R}^{p \times p}$.
  Compute $\tilde{\mathscr{A}} = \mathscr{A} \times_3 M$
  for $i = 1$ to p **do**
  Compute $\tilde{\mathscr{L}}(:,:,i) = lowerdiag(\tilde{\mathscr{A}}(:,:,i))$, $\tilde{\mathscr{U}}(: . :, i) = \tilde{\mathscr{A}}(:,:,i) - \tilde{\mathscr{L}}(:,:,i) - diag(\tilde{\mathscr{A}}(:,:,i))$
  Compute $\tilde{\mathscr{T}}(:,:,i) = -(\tilde{\mathscr{L}})^{-1}(:,:,i)\tilde{\mathscr{U}}(: . :, i)$ and $\mathscr{C}(:,1,i) = (\tilde{\mathscr{L}})^{-1}(:,:,i)\mathscr{B}(:,1,i)$
  Initial guess $\tilde{\mathscr{X}}^0(:,1,i)$
  for $s = 1$ to MAX **do**
  $\tilde{\mathscr{X}}^s(:,1,i) = \tilde{\mathscr{T}}(:,:,i)\tilde{\mathscr{X}}^{s-1}(:,1,i) + \mathscr{C}(:,1,i)$
  if $\|\tilde{\mathscr{X}}^s(:,1,i) - \tilde{\mathscr{X}}^0(:,1,i)\| \leq \varepsilon$ **then**
  break
  if
  $\tilde{\mathscr{X}}^0(:,1,i) \leftarrow \tilde{\mathscr{X}}^s(:,1,i)$
  end for
  end for
  Compute $\mathscr{X}^s = \tilde{\mathscr{X}}^s \times_3 M^{-1}$
  return $\mathscr{X}^s$
end procedure

Higher order Gauss-Seidel method

Table: Comparison analysis of CPU-time, residual errors for Gauss-Seidel method for different order tensors and matrices with $\varepsilon = 10^{-10}$

Size of $\mathscr{A}$	IT^M	MT^M	Size of A	IT	MT
$100 \times 100 \times 400$	15	0.21	2000×2000	16	8.33
$200 \times 200 \times 400$	15	0.60	4000×4000	17	389.57
$300 \times 300 \times 400$	15	0.76	6000×6000	17	1227.65
$400 \times 400 \times 400$	15	1.12	8000×8000	17	3506.91
$500 \times 500 \times 400$	15	1.46	10000×10000	17	5371.06

OUTLINE

Two-step Alternating iterative Scheme

- Let $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ be two splittings of $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. Then:

$$\mathscr{Y}^{k+1} = \mathscr{F}^{-1} *_M \mathscr{G} *_M \mathscr{X}^k + \mathscr{F}^{-1} *_M \mathscr{B} \quad (1)$$

$$\mathscr{X}^{k+1} = \mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{Y}^{k+1} + \mathscr{K}^{-1} *_M \mathscr{B} \quad (2)$$

Two-step Alternating iterative Scheme

- Let $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ be two splittings of $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. Then:

$$\mathscr{Y}^{k+1} = \mathscr{F}^{-1} *_M \mathscr{G} *_M \mathscr{X}^{k} + \mathscr{F}^{-1} *_M \mathscr{B} \tag{1}$$

$$\mathscr{X}^{k+1} = \mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{Y}^{k+1} + \mathscr{K}^{-1} *_M \mathscr{B} \tag{2}$$

- By simplifying the iterative schemes (1) and (2) we have

$$\mathscr{X}^{k+1} = \mathscr{H} *_M \mathscr{X}^{k} + \mathscr{C} *_M \mathscr{B}, \tag{3}$$

where $\mathscr{H} = \mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{F}^{-1} *_M \mathscr{G}$ and $\mathscr{C} = \mathscr{K}^{-1} \mathscr{L} *_M \mathscr{F}^{-1} + \mathscr{K}^{-1}$.

Two-step Alternating iterative Scheme

- Let $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ be two splittings of $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. Then:

$$\mathscr{Y}^{k+1} = \mathscr{F}^{-1} *_M \mathscr{G} *_M \mathscr{X}^{k} + \mathscr{F}^{-1} *_M \mathscr{B} \tag{1}$$

$$\mathscr{X}^{k+1} = \mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{Y}^{k+1} + \mathscr{K}^{-1} *_M \mathscr{B} \tag{2}$$

- By simplifying the iterative schemes (1) and (2) we have

$$\mathscr{X}^{k+1} = \mathscr{H} *_M \mathscr{X}^{k} + \mathscr{C} *_M \mathscr{B}, \tag{3}$$

where $\mathscr{H} = \mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{F}^{-1} *_M \mathscr{G}$ and $\mathscr{C} = \mathscr{K}^{-1} \mathscr{L} *_M \mathscr{F}^{-1} + \mathscr{K}^{-1}$.

Definition

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$. A splitting $\mathscr{A} = \mathscr{F} - \mathscr{G}$ is called

- regular splitting of $\mathscr{A}$ if $\mathscr{F}^{-1} \geq 0$ and $\mathscr{G} \geq 0$.
- weak regular splitting of $\mathscr{A}$ if $\mathscr{F}^{-1} \geq 0$ and $\mathscr{F}^{-1} *_M \mathscr{G} \geq 0$.

Two-step Alternating iterative Scheme

The convergence and comparison theorem of the proposed iteration scheme which we proved as the followings:

Theorem (1)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $\mathscr{A}^{-1} \geq 0$. If $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ are two weak regular splittings of $\mathscr{A}$ then $\rho(\mathscr{H}) = \rho(\mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{F}^{-1} *_M \mathscr{G}) < 1$.

Two-step Alternating iterative Scheme

The convergence and comparison theorem of the proposed iteration scheme which we proved as the followings:

Theorem (1)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $\mathscr{A}^{-1} \geq 0$. If $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ are two weak regular splittings of $\mathscr{A}$ then $\rho(\mathscr{H}) = \rho(\mathscr{K}^{-1} *_M \mathscr{L} *_M \mathscr{F}^{-1} *_M \mathscr{G}) < 1$.

Theorem (2)

Let $\mathscr{A} \in \mathbb{R}^{m\times n\times p}$ and $\mathscr{A}^{-1} \geq 0$. If $\mathscr{A} = \mathscr{F} - \mathscr{G} = \mathscr{K} - \mathscr{L}$ are two regular splittings of $\mathscr{A}$, then

$$\rho(\mathscr{H}) \leq \min\{\rho(\mathscr{F}^{-1} *_M \mathscr{G}), \rho(\mathscr{K}^{-1} *_M \mathscr{L})\} < 1.$$

Numerical Examples

TABLE: Comparison analysis of CPU-time, residual errors for different order tensors and matrices with $\varepsilon = 10^{-10}$

Size of $\mathscr{A}$	MT	$\|\mathscr{A} *_M \mathscr{X} - \mathscr{B}\|$	Order of A	MT	$\|AX - b\|$
$100 \times 100 \times 400$	0.19	$1.8e^{-11}$	2000	7.56	$2.1e^{-10}$
$200 \times 200 \times 400$	0.53	$3.4e^{-11}$	4000	295.5	$3.5e^{-09}$
$300 \times 300 \times 400$	0.95	$4.2e^{-11}$	6000	1175.3	$6.4e^{-09}$
$400 \times 400 \times 400$	1.27	$5.9e^{-11}$	8000	2965.9	$2.7e^{-09}$

NUMERICAL EXAMPLES

TABLE: Comparison analysis of CPU-time two-step against one step method for different order tensors

Size of $\mathscr{A}$	$\frac{\text{IT}}{\text{two-step}}$	$\frac{\text{MT}}{\text{two-step}}$	$\frac{\text{IT}}{\text{one-step}}$	$\frac{\text{MT}}{\text{one-step}}$
$100 \times 100 \times 400$	78	0.31	86	0.42
$200 \times 200 \times 400$	84	0.67	101	0.97
$300 \times 300 \times 400$	89	0.98	113	1.76
$400 \times 400 \times 400$	109	1.76	149	3.14

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THANK YOU